Stresses and Rate of Twist in Single-Cell Thin-Walled Beams with Anisotropic Walls

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The classical theory of elastic thin-walled beams of (assumedly) nondeformable cross section, familiar to aerospace structures engineers, is generalized to the case in which the walls are anisotropic. The assumed anisotropy is such as to allow coupling between cross-sectional shear flow and longitudinal strain and, reciprocally, between shear strain and longitudinal stress. The theory and several illustrative applications show that anisotropy of the walls can, as expected, lead to coupling phenomena not present in the classical theory, e.g., twist due to bending mements, bending due to torque, twist due to tension, and extension due to torque, thus confirming the well-known possibility of "tailoring" elastic behavior through the use of laminated composites. It is also shown that anisotropy of the walls can lead to a nonconstant rate of twist in uniform beams even if the cross-sectional torque and shears are constant. Another interesting outcome of the present theory is its prediction that the shear flows are independent of the coupling constant in the constitutive equations, i.e., the constant that defines the longitudinal strain due to shear flow and the shear strain due to longitudinal stress, and also independent of the elastic constant relating shear strain to shear flow.

I. Introduction

THE calculation of shear flows and rate of twist in uniform thin-walled elastic beams (single-cell or multicell) is a basic problem in the stress analysis of aerospace structures. Its solution by simple methods is treated in many books (e.g., Refs. 1-3). In the case of uniform torsion, the simple method is attributed to Bredt[4] and involves a formula of the form

$$\frac{d\phi}{dz} = \frac{1}{2A}\int \frac{q\, ds}{Gt} \tag{1}$$

for evaluating the rate of twist $d\phi/dz$ of a cell. For combined bending and torsion, a generalization of Bredt's method is employed: longitudinal cuts are first assumed, of sufficient number to destroy the closedness of the cross section. "Primary shear flows" are then computed by means of the "VQ/I" formula (or its generalization, if the cross section is unsymmetric), and unknown correction shear flows, one for each cell, are superimposed on the primary shear flows to compensate for the cuts. The correction shear flows are determined from an equation of static equivalence (the external loads acting on a cross section must have the same moment about a longitudinal axis as the cross-sectional shear flows), supplemented (if the beam has more than one cell) by equations of compatible rates of twist of the cells. The latter are statements to the effect that the rates of twist of the individual cells, as evaluated from Eq. (1), are equal.

In the case of combined bending and torsion, the method just described is inexact for a number of reasons, among which are the following: 1) although the stresses obtained satisfy equilibrium, the strains implied by them are not necessarily compatible; 2) the method assumes that the shape of the cross sections is preserved, thereby neglecting Poisson's ratio effects; and 3) the method assumes that the cross-sectional normal stresses are given by the engineer's theory of bending (through the "My/I" formula, or its generalization in the case of unsymmetrical bending), thus neglecting "shear lag" and "bending stress due to torsion," and making it usually inappropriate for stubby beams with noncompact cross sections.

In spite of its inexactness, this simple and classical method enjoys considerable popularity, even in the present finite-element age. One must, therefore, assume that it has had some measure of success. At the same time, inasmuch as this is also the age of composites, the method does not have the generality called for by much of the present-day construction. Implicit in the method is the assumption that the walls are isotropic, or at least specially orthotropic with one of the axes of elastic symmetry parallel to the longitudinal axis of the beam. Many present-day wing and helicopter blade designs employ laminated composites that are anisotropic. By design (e.g., for tailoring purposes), they may be anisotropic in such a way that normal stress at a point of the cross section tends to produce shear strain as well as longitudinal strain, whereas cross-sectional shear flow tends to produce longitudinal strain along with shear strain. Such walls are neither isotropic nor specially orthotropic, and the classical Bredt method and its extension are not applicable, therefore, to the beams containing them.

The purpose of the present paper is to generalize the classical method so that it will apply to thin-walled beams with anisotropic walls possessing the kind of elastic coupling described above and, therefore, to thin-walled beams with arbitrary laminated-composite walls. The spirit of the classical method is retained in this generalization. Therefore, it will possess inconsistencies and shortcomings similar to those cited above for the classical method. It is believed, however, that it will also possess the same kind of utility as the classical method when applied to geometries for which it is appropriate, e.g., to relatively slender configurations like helicopter blades. For simplicity, only single-cell beams are considered. The extension to multicell beams offers no conceptual difficulty. As in the case of the classical theory, such an extension requires that the equation of static equivalence be supplemented by equations of compatible (equal) rates of twist of the several cells.

Both in spirit and results the present method is very similar to that developed by Mansfield in the first five sections of Ref. 5, although the present author did not become aware of Mansfield's work until after his own analysis was completed. However, the present derivation is thought to be more

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straightforward than Mansfield's. (Rehfield has referred to Mansfield's theoretical development as "sometimes hard to follow."[6]) It also may be slightly more rigorous, inasmuch as two aspects of beam behavior established by postulate in Ref. 5 are arrived at by deduction in the present paper. They are 1) the linearity of ϵ_z with respect ot z in Mansfield's Eq. (30), and 2) the constancy of the shear flow N_{zs} with respect to z, postulated just after Eq. (31). For the above reasons, the present author does not believe that his analysis is obviated by the earlier publication of Mansfield's.

II. Geometry and Loading

We consider thin-walled elastic beams, or segments of such beams, that are essentially cylindrical tubes with no taper or initial twist (Fig. 1) and with wall properties (thickness and elastic constants) that may vary from point to point around the section but are uniform in the longitudinal direction. That is, referring to the coordinate system in Fig. 1, we assume that the wall properties may vary with s but are independent of z, where s is a distance coordinate measured along the periphery of the cross section from some reference line aa, and z defines the location of the cross section.

A point in the wall at a given cross section can be identified by its coordinate s or, alternatively, by its coordinates $x(s)$ and $y(s)$ in the arbitrary Cartesian coordinate system shown in Fig. 1. All of the cross sections are visualized as having such a coordinate system, with their origins 0 along a straight axis parallel to the generators of the tube, their x axes all lying in the same plane, and their y axes similarly coplanar.

The beam or beam segment is assumed to be subjected to an equilibrium system of external loads consisting only of forces and couples applied to the end cross sections ($z=0$ and $z=L$). This loading will give rise to cross-sectional loadings that, at any cross section, can be represented by two shears V_x and V_y, a thrust P, two bending mements M_x and M_y, and a torque T, as shown in Fig. 1.

It should be noted that the assumed loading does not include loads applied between the end cross sections, i.e., in the region $0<x<L$. Consequently, the cross-sectional shears V_x and V_y and the cross-sectional torques T and tensions P will be constant (independent of z) along the length L of the beam or beam segment. In an actual beam there very well may be distributed forces and couples (arising, for example, from aerodynamic and centrifugal loadings). In order for the present threory to apply to such beams, any distributed loadings will have to be approximated by concentrated forces and couples at discrete cross sections. Any successive two such cross sections will delimit a beam segment loaded only at its end cross sections, and to which the present theory is, therefore, applicable. The distance between the two successive cross sections will establish the length L of the segment. The cross-sectional loads V_x, V_y, T, and P will, in general, be

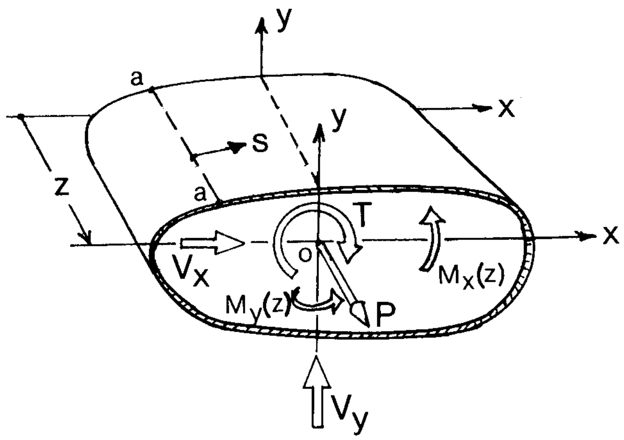

Fig. 1 Coordinate system and cross-sectional loads.

constant within the segment but exhibit jumps from one segment to the next. If a beam is nonuniform, a similar artifice can be employed in order to make the present theory applicable; i.e., the nonuniform beam can be replaced by a piecewise uniform approximation.

Finally, we note that whereas V_x, V_y, T, and P are independent of z within any beam or beam segment, the other two cross-sectional loads M_x and M_y will vary linearly with z in consequence of the equilibrium equations

$$\mathrm{d}M_x/\mathrm{d}z = -V_y = \text{const} \tag{2a}$$

$$\mathrm{d}M_y/\mathrm{d}z = -V_x = \text{const} \tag{2b}$$

These lead to the following expressions for M_x and M_y as functions of z:

$$M_x(z) = M_x(L) + (L-z)V_y = \frac{z}{L}M_x(L) + \left(1-\frac{z}{L}\right)M_x(0) \tag{3a}$$

$$M_y(z) = M_y(L) + (L-z)V_x = \frac{z}{L}M_y(L) + \left(1-\frac{z}{L}\right)M_y(0) \tag{3b}$$

Later, we shall find it convenient to introduce the symbols $\bar{M}_x$ and $\bar{M}_y$ to stand for the average values of M_x and M_y over the length L, i.e., the values of M_x and M_y at $z=L/2$.

III. Constitutive Equations

The walls are assumed to be thin enough to be regarded as membranes in plane stress. In addition, the normal stress resultant N_s in the s direction is assumed to be negligible, so that the state of stress at any point can be described by a shear flow q and a longitudinal "tension flow" N (both having units of force per unit width) as shown in Fig. 2. These will produce a shear strain γ, a longitudinal strain ϵ, and a transverse strain ϵ_s, the latter playing no role in the present analysis. The strains ϵ and γ are assumed to be related to N and q by equations of the form

$$\epsilon = \alpha_1 N + \alpha_2 q \tag{4a}$$

$$\gamma = \alpha_2 N + \alpha_4 q \tag{4b}$$

where α_1, α_2, and α_4 are elastic constants. In the derivations that follow, it will be convenient to use the following hybrid form of Eqs. (4):

$$N = \beta_1\epsilon + \beta_2 q \tag{5a}$$

$$\gamma = -\beta_2\epsilon + \beta_4 q \tag{5b}$$

where the various β are new elastic constants related as follows to the various α:

$$\beta_1 = 1/\alpha_1 \tag{6a}$$

$$\beta_2 = -\alpha_2/\alpha_1 \tag{6b}$$

$$\beta_4 = \alpha_4 - (\alpha_2^2/\alpha_1) \tag{6c}$$

$$\alpha_1 = 1/\beta_1 \tag{6d}$$

$$\alpha_2 = -\beta_2/\beta_1 \tag{6e}$$

$$\alpha_4 = \beta_4 + (\beta_2^2/\beta_1) \tag{6f}$$

If the full set of anisotropic plane-stress constitutive equations at a point in the wall is known in the form

$$\epsilon = S_{11}N + S_{12}N_s + S_{14}q \tag{7a}$$

$$\epsilon_s = S_{12}N + S_{22}N_s + S_{24}q \tag{7b}$$

$$\gamma = S_{14}N + S_{24}N_s + S_{44}q \tag{7c}$$

then, by setting N_s equal to zero and comparing the resulting equations with Eqs. (4), it is seen that

$$\alpha_1 = S_{11} \tag{8a}$$

$$\alpha_2 = S_{14} \tag{8b}$$

$$\alpha_4 = S_{44} \tag{8c}$$

By virtue of their roles in Eqs. (4) and (5), α_2 and β_2 may be referred to as the coupling elastic constants.

IV. Analysis

Preliminary Considerations

The differential equations of equilibrium for the infinitesimal element in Fig. 2 are

$$\frac{\partial q}{\partial z} = 0 \tag{9a}$$

$$\frac{\partial q}{\partial s} + \frac{\partial N}{\partial z} = 0 \tag{9b}$$

From Eq. (9a) it follows that $q = q(s)$, i.e., the shear flow is a function of s only. Integration of Eq. (9b) gives

$$q(s) = q_0 - \int_0^s \left(\frac{\partial N}{\partial z}\right) \mathrm{d}s \tag{10}$$

where q_0 is the cross-sectional (and longitudinal) shear flow at $s = 0$. Figure 3 provides a visual aid for the interpretation of Eq. (10). It will be noted that the two terms on the right side of Eq. (10) correspond, respectively, to the correction shear flow and the VQ/I primary shear flow of the classical theory of thin-walled beams.

In the classical theory of isotropic homogeneous thin-walled beams, the cross-sectional normal stress is assumed to vary linearly with x and y. The corresponding assumption to be employed here is that the longitudinal strain ϵ is a linear function of x and y, i.e.,

$$\epsilon = \epsilon_0 - yK_x - xK_y \tag{11}$$

where ϵ_0, K_x, and K_y are as yet unknown functions of z. Substitution of Eq. (11) into Eq. (5a) gives

$$N = \beta_1(\epsilon_0 - yK_x - xK_y) + \beta_2 q \tag{12}$$

whence

$$\frac{\partial N}{\partial z} = \beta_1(\epsilon_0' - yK_x' - xK_y') \tag{13}$$

where the primes denote differentiation with respect to z, and we have taken note of Eq. (9a). Equation (10) then gives

$$q(s) = q_0 - \epsilon_0' \, a_1(s) + K_x' \, a_2(s) + K_y' \, a_3(s) \tag{14}$$

where

$$a_1(s) = \int_0^s \beta_1 \, \mathrm{d}s \tag{15a}$$

$$a_2(s) = \int_0^s y\beta_1 \, \mathrm{d}s \tag{15b}$$

$$a_3(s) = \int_0^s x\beta_1 \, \mathrm{d}s \tag{15c}$$

We note for later use that substitution of Eqs. (11) and (14) into Eq. (5b) gives

$$\gamma = -\beta_2(\epsilon_0 - yK_x - xK_y) + \beta_4[q_0 - \epsilon_0' a_1(s) + K_x' a_2(s) + K_y' a_3(s)] \tag{16}$$

Equations of Static Equivalence

At any cross section z, the tensions N must be statically equivalent to the cross-sectional loadings P, M_x, and M_y. Therefore, $P = \oint N \, \mathrm{d}s$, $M_x = -\oint yN \, \mathrm{d}s$, and $M_y = -\oint xN \, \mathrm{d}s$, where the integrations are entirely around the cross section starting from $s = 0$. Using Eq. (12) to eliminate N in these integrals, we obtain

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{Bmatrix} \epsilon_0 \\ K_x \\ K_y \end{Bmatrix} = \begin{Bmatrix} P - Q_1 \\ -M_x - Q_2 \\ -M_y - Q_3 \end{Bmatrix} \tag{17}$$

where

$$b_{11} = \oint \beta_1 \, \mathrm{d}s, \qquad b_{12} = -\oint y\beta_1 \, \mathrm{d}s, \qquad b_{13} = -\oint x\beta_1 \, \mathrm{d}s \tag{18a}$$

$$b_{21} = \oint y\beta_1 \, \mathrm{d}s, \quad b_{22} = -\oint y^2\beta_1 \, \mathrm{d}s, \quad b_{23} = -\oint yx\beta_1 \, \mathrm{d}s \tag{18b}$$

$$b_{31} = \oint x\beta_1 \, \mathrm{d}s, \quad b_{32} = -\oint xy\beta_1 \, \mathrm{d}s, \quad b_{33} = -\oint x^2\beta_1 \, \mathrm{d}s \tag{18c}$$

$$Q_1 = \oint q\beta_2 \, \mathrm{d}s, \qquad Q_2 = \oint yq\beta_2 \, \mathrm{d}s, \qquad Q_3 = \oint xq\beta_2 \, \mathrm{d}s \tag{19}$$

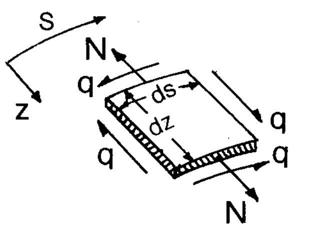

Fig. 2 Assumed state of stress on infinitesimal element of wall.

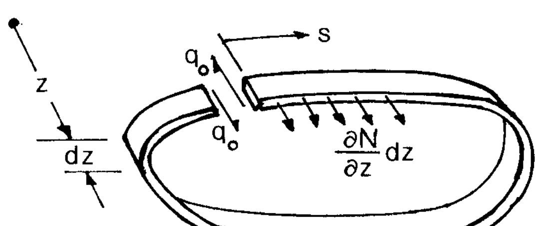

Fig. 3 Visual aid for Eq. (10).

Letting $[a_{ij}]$ be the inverse of the matrix $[b_{ij}]$, we have from Eq. (17)

$$\begin{Bmatrix} \epsilon_0 \\ K_x \\ K_y \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} P - Q_1 \\ -M_x - Q_2 \\ -M_y - Q_3 \end{Bmatrix} \tag{20}$$

Because of the presence of M_x and M_y in Eq. (20), ϵ_0, K_x, and K_y will be linear functions of z. Differentiating Eq. (20) with respect to z while taking note of Eq. (2), Eq. (9a), and the constancy of P, we obtain

$$\begin{Bmatrix} \epsilon_0' \\ K_x' \\ K_y' \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} 0 \\ V_y \\ V_x \end{Bmatrix} \tag{21}$$

from which it is evident that ϵ_0', K_x', and K_y' are independent of z.

The shear flows $q(s)$ must be statically equivalent to the torque T. That is, $T = \oint p(s) q(s)\, ds$, where $p(s)$ is the perpendicular distance from 0 to the tangent to the cross section at the point defined by s (see Fig. 4). Using Eq. (14) to eliminate $q(s)$ in this integral, then solving for q_0, we obtain

$$q_0 = \left(\frac{1}{2A}\right)(T + \epsilon_0' a_4 - K_x' a_5 - K_y' a_6) \tag{22}$$

where

$$A = \frac{1}{2}\oint p(s)\, ds = \text{area enclosed by the cell} \tag{23}$$

$$a_4 = \oint p(s) a_1(s)\, ds \tag{24a}$$

$$a_5 = \oint p(s) a_2(s)\, ds \tag{24b}$$

$$a_6 = \oint p(s) a_3(s)\, ds \tag{24c}$$

Inasmuch as ϵ_0', K_x', and K_y' are known from Eq. (21), Eq. (22) completely defines q_0; then Eq. (14) completely defines $q(s)$. Furthermore, the constancy of ϵ_0', K_x', and K_y' ensures that q_0 and $q(s)$ are independent of z, as required by Eq. (9a). Tracing through the steps leading to $q(s)$, we also note the following interesting results regarding the shear flows: 1) they depend on $\beta_1(s)$, but not at all on $\beta_4(s)$ or on the coupling elastic constant $\beta_2(s)$; 2) although they depend on $\beta_1(s)$, the dependence is only on the form of that function, not on its amplitude [that is, if $\beta_1(s)$ is represented as $\beta_{1_0}F(s)$, where β_{1_0} is an amplitude constant and $F(s)$ a fixed function of s, then $q(s)$ turns out to be independent of β_{1_0}]; and 3) by virtue of Eqs. (6b) and (8), the shear flows' independence of β_2 and β_4 implies their independence of α_2 and α_4, likewise their independence of S_{14} and S_{44}.

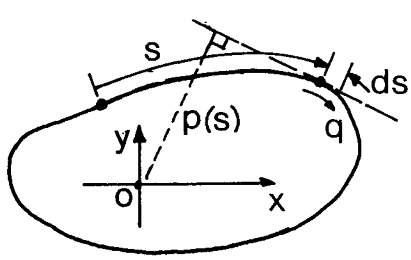

Fig. 4 Perpendicular distance $p(s)$ associated with any point (s) of the cross section.

With $q(s)$ now known through Eq. (14), the integrals Q_1, Q_2, and Q_3, defined by Eq. (19), can be evaluated. The result is

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \end{bmatrix} \begin{Bmatrix} q_0 \\ \epsilon_0' \\ K_x' \\ K_y' \end{Bmatrix} \tag{25}$$

where

$$c_{11} = \oint \beta_2\, ds, \qquad c_{12} = -\oint a_1(s)\beta_2\, ds$$

$$c_{13} = \oint a_2(s)\beta_2\, ds, \qquad c_{14} = \oint a_3(s)\beta_2\, ds$$

$$c_{21} = \oint y\beta_2 ds, \qquad c_{22} = -\oint y a_1(s)\beta_2\, ds$$

$$c_{23} = \oint y a_2(s)\beta_2\, ds, \qquad c_{24} = \oint y a_3(s)\beta_2\, ds$$

$$c_{31} = \oint x\beta_2\, ds, \qquad c_{32} = -\oint x a_1(s)\beta_2\, ds$$

$$c_{33} = \oint x a_2(s)\beta_2\, ds, \qquad c_{34} = \oint x a_3(s)\beta_2\, ds \tag{26}$$

Equation (20), in conjunction with Eqs. (25), (22), and (21), now completely defines ϵ_0, K_x, K_y.

Rate of Twist

In the classical theory of thin-walled beams, the rate of twist of a cell is related to the shear strains in the walls of the cell by the following more fundamental form of Eq. (1):

$$\frac{d\phi}{dz} = \frac{1}{2A}\oint \gamma(s)\, ds \tag{27}$$

in which $\phi(z)$ is the angle of twist. Inasmuch as Eq. (27) is based only on geometric considerations (not on material properties), it remains valid in the present context; and with $\gamma(s)$ eliminated via Eq. (16), it gives

$$\frac{d\phi}{dz} = \frac{1}{2A}(-\epsilon_0 c_{11} + K_x c_{21} + K_y c_{31} + q_0\, d_1 + \epsilon_0'\, d_2 + K_x'\, d_3 + K_y'\, d_4) \tag{28}$$

where

$$d_1 = \oint \beta_4\, ds \tag{29a}$$

$$d_2 = -\oint a_1(s)\beta_4\, ds \tag{29b}$$

$$d_3 = \oint a_2(s)\beta_4\, ds \tag{29c}$$

$$d_4 = \oint a_3(s)\beta_4\, ds \tag{29d}$$

Since ϵ_0, K_x, K_y, ϵ_0', K_x', K_y', and q_0 have already been determined, Eq. (28) completely defines $d\phi/dz$. In general, it will be a linear function of z because ϵ_0, K_x, and K_y are linear functions of z, while q_0, ϵ_0', K_x', and K_y' are constants. By integration of Eq. (28), the following formula is obtained for the average rate of twist over the length L, which is also the local rate of twist at $z = L/2$:

$$(1/L)\,[\phi(L) - \phi(0)] = (1/2A)(-\bar{\epsilon}_0 c_{11} + \bar{K}_x c_{21} + \bar{K}_y c_{31} + q_0\, d_1 + \epsilon_0' d_2 + K_x' d_3 + K_y' d_4) \tag{30}$$

where $\bar{\epsilon}_0$, $\bar{K}_x$, and $\bar{K}_y$ are the average values of ϵ_0, K_x, and K_y, i.e., their values at $z = L/2$, obtained by substituting $M_x = \bar{M}_x = M_x(L/2)$ and $M_y = \bar{M}_y = M_y(L/2)$ into the right side of Eq. (20).

Although Eq. (28) is quite adequate for computational purposes, a formula giving $d\phi/dz$ more explicitly in terms of loads can afford more insight. Such a formula is obtained by using Eqs. (20) and (25) to eliminate ϵ_0, K_x, and K_y in Eq. (28); then Eq. (22) to eliminate q_0; and finally Eq. (21) to eliminate ϵ_0', K_x', and K_y'. The result is

$$\frac{d\phi}{dz} = \frac{1}{2A}\left[ST + (SA_R + D_R - C_R A_S C_S) A_S \begin{Bmatrix} 0 \\ V_y \\ V_x \end{Bmatrix} + C_R A_S \begin{Bmatrix} P \\ -M_x \\ -M_y \end{Bmatrix} \right] \tag{31}$$

in which A_S and C_S are square matrices; A_R, C_R, and D_R are row matrices; and S is a scalar. Their definitions are

$A_S = [a_{ij}]$ matrix of Eq. (20)

$C_S = [c_{ij}]$ matrix of Eq. (25) with first column omitted

$$A_R = [a_4 \quad -a_5 \quad -a_6]$$

$$C_R = [-c_{11} \quad c_{21} \quad c_{31}]$$

$$D_R = [d_2 \quad d_3 \quad d_4]$$

$$S = (1/2A)(d_1 - C_R A_S C_C) \tag{32}$$

where C_c is the column matrix $[c_{11} \quad c_{21} \quad c_{31}]^T$. The average rate of twist over the length L is obtained by replacing M_x and M_y in Eq. (31) by $\bar{M}_x$ and $\bar{M}_y$, respectively, with the result

$$\frac{1}{L}[\phi(L) - \phi(O)] = \frac{1}{2A}\left[ST + (SA_R + D_R - C_R A_S C_S) A_S \begin{Bmatrix} 0 \\ V_y \\ V_x \end{Bmatrix} + C_R A_S \begin{Bmatrix} P \\ -\bar{M}_x \\ -\bar{M}_y \end{Bmatrix} \right] \tag{33}$$

Shear Center

In the classical linear theory of uniform thin-walled beams with isotropic walls, the rate of twist $d\phi/dz$ depends only on the torque T and the shears V_x and V_y, and it is independent of z as long as T, V_x, and V_y are independent of z. Because of these characteristics, a point (the shear center) can be found such that a cross-sectional shear passing through that point will produce no rate of twist. This point is a geometric property of the cross section, independent of the loading.

In the present theory, on the other hand, Eq. (31) shows that the local rate of twist depends not only on T, V_x, and V_y, but also on P, M_x, and M_y. Furthermore, since M_x and M_y inevitably vary with z when any shear is present, it follows that the rate of twist cannot be made zero everywhere along the length L by proper choice of the line of action of the cross-sectional shear. At best, one can make the average rate of twist, as given by Eq. (33), zero by proper selection of the line of action of a given shear. This line of action, however, will not be a property of the cross section; it will depend on the values of P, $\bar{M}_x$, and $\bar{M}_y$ that are present with the shear.

From these observations, it appears that a shear center in the usual sense does not exist in the present context. However, one can formulate a shear center definition that is nearly analogous to that of the classical theory by defining the shear center as that point through which a cross-sectional shear must pass in order that the average rate of twist as given by Eq. (33) is zero when P, $\bar{M}_x$, and $\bar{M}_y$ are zero. The shear center defined in this way will be a property of the cross section. However, its usefulness is questionable.

Calculation Procedure

At this point, all of the pertinent formulas have been developed. The following sequence of calculations will yield the cross-sectional stress quantities $q(s)$ and $N(s,z)$, the strains $\epsilon(s,z)$ and $\gamma(s,z)$, and the rate of twist $d\phi/dz$:

1) Evaluate $a_1(s)$, $a_2(s)$, $a_3(s)$ from Eqs. (15).
2) Determine the b_{ij} from Eqs. (18).
3) Invert the $[b_{ij}]$ matrix to get the a_{ij}.
4) Use Eq. (23) or any other convenient method to evaluate A.
5) Compute a_4, a_5, a_6 from Eqs. (24).
6) Compute ϵ_0', K_x', K_y' from Eq. (21).
7) Determine q_0 from Eq. (22), then $q(s)$ from Eq. (14).
8) Evaluate d_1, d_2, d_3, d_4 from Eqs. (29).
9) Evaluate the c_{ij} from Eqs. (26).
10) Determine Q_1, Q_2, Q_3 from Eqs. (25).
11) Determine ϵ_0, K_x, K_y from Eq. (20).
12) Find the tensions $N(s,z)$ from Eq. (12).
13) Find the longitudinal strains $\epsilon(s,z)$ from Eqs. (4a) or (11).
14) Find the shear strains $\gamma(s,z)$ from Eqs. (4b), (5b), or (16).
15) Obtain the rate of twist $d\phi/dz$ from Eqs. (28) or (31) and/or the average rate of twist over the length L from Eqs. (30) or (33).

It will be noted that steps 1 through 8 do not involve the coupling constant β_2, which implies that the shear flows $q(s)$ do not depend on that constant (this independence has been noted earlier). However, steps 9 through 15 do involve β_2; therefore, $N(s,z)$, $\epsilon(s,z)$, $\gamma(s,z)$, and $d\phi/dz$ are affected by it.

V. Illustrative Applications

A. Homogeneous Circular Tube

We consider a cantilever beam of circular cross section (Fig. 5) in which the elastic constants, β_1, β_2, β_4 are independent of

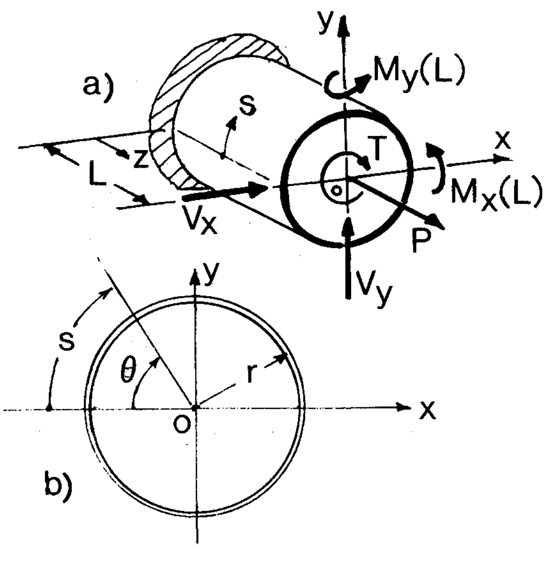

Fig. 5 Circular tube: a) loading, b) cross section.

both s and z, as in the case of a spiral-like wall construction. The tube is subjected to a full complement of end loads, viz., V_x, V_y, T, P, $M_x(L)$, and $M_y(L)$, producing cross-sectional loads of V_x, V_y, T, P, $M_x = M_x(L) + (L-z)V_y$, and $M_y = M_y(L) + (L-z)V_x$. Following the calculation steps listed above and noting that $p(s) = \text{const} = r$, $ds = r\,d\theta$, $x = -r\cos\theta$, and $y = r\sin\theta$, we obtain the results shown below. [It will be noted in this and the subsequent applications that notational rigor is frequently sacrificed for the sake of simplicity. For example, the quantity named a_2 is found as a function of θ but still denoted $a_2(s)$.]

Step 1:

$$a_1(s) = \beta_1 s = \beta_1 r\theta$$

$$a_2(s) = \beta_1\int_0^\theta (r\sin\theta)\,r\,d\theta = \beta_1 r^2(1-\cos\theta)$$

$$a_3(s) = \beta_1\int_0^\theta (-r\cos\theta)\,r\,d\theta = -\beta_1 r^2\sin\theta$$

Step 2:

$$b_{11} = 2\pi r\beta_1$$

$$b_{22} = -\beta_1\int_0^{2\pi} (r\sin\theta)^2\,r\,d\theta = -\pi r^3\beta_1$$

$$b_{33} = -\beta_1\int_0^{2\pi} (-r\cos\theta)^2\,r\,d\theta = -\pi r^3\beta_1$$

$$b_{ij} = 0 \qquad \text{for } i \neq j$$

Step 3:

$$a_{11} = (2\pi r\beta_1)^{-1}$$

$$a_{22} = a_{33} = (-\pi r^3\beta_1)^{-1}$$

$$a_{ij} = 0 \qquad \text{for } i \neq j$$

Step 4:

$$A = \pi r^2$$

Step 5:

$$a_4 = \int_0^{2\pi} r(\beta_1 r\theta)\,r\,d\theta = 2\pi^2 r^3\beta_1$$

$$a_5 = \int_0^{2\pi} r[\beta_1 r^2(1-\cos\theta)]\,r\,d\theta = 2\pi r^4\beta_1$$

$$a_6 = \int_0^{2\pi} r(-\beta_1 r^2\sin\theta)\,r\,d\theta = 0$$

Step 6:

$$\epsilon_0' = 0$$

$$K_x' = a_{22}V_y = (-\pi r^3\beta_1)^{-1}V_y$$

$$K_y' = a_{33}V_x = (-\pi r^3\beta_1)^{-1}V_x$$

Step 7:

$$q_0 = (T/2\pi r^2) + (V_y/\pi r)$$

$$q(s) = (T/2\pi r^2) + (1/\pi r)(V_y\cos\theta + V_x\sin\theta)$$

Step 8:

$$d_1 = 2\pi r\beta_4$$

$$d_2 = -\beta_4\int_0^{2\pi} (\beta_1 r\theta)\,r\,d\theta = -2\pi^2 r^2\beta_1\beta_4$$

$$d_3 = \beta_4\int_0^{2\pi} \beta_1 r^2(1-\cos\theta)\,r\,d\theta = 2\pi r^3\beta_1\beta_4$$

$$d_4 = \beta_4\int_0^{2\pi} (-\beta_1 r^2\sin\theta)\,r\,d\theta = 0$$

Step 9:

$$c_{11} = 2\pi r\beta_2$$

$$c_{21} = \beta_2\oint y\,ds = 0$$

$$c_{31} = \beta_2\oint x\,ds = 0$$

$$c_{12} = -\int_0^{2\pi} (\beta_1 r\theta)\,\beta_2\,r\,d\theta = -2\pi^2 r^2\beta_1\beta_2$$

$$c_{22} = -\int_0^{2\pi} (r\sin\theta)(\beta_1 r\theta)\beta_2\,r\,d\theta = 2\pi r^3\beta_1\beta_2$$

$$c_{32} = -\int_0^{2\pi} (-r\cos\theta)(\beta_1 r\theta)\,\beta_2\,r\,d\theta = 0$$

$$c_{13} = \int_0^{2\pi} \beta_1 r^2(1-\cos\theta)\beta_2\,r\,d\theta = 2\pi r^3\beta_1\beta_2$$

$$c_{23} = \int_0^{2\pi} (r\sin\theta)\beta_1 r^2(1-\cos\theta)\beta_2\,r\,d\theta = 0$$

$$c_{33} = \int_0^{2\pi} (-r\cos\theta)\beta_1 r^2(1-\cos\theta)\beta_2\,r\,d\theta = \pi r^4\beta_1\beta_2$$

$$c_{14} = \int_0^{2\pi} (-\beta_1 r^2\sin\theta)\beta_2\,r\,d\theta = 0$$

$$c_{24} = \int_0^{2\pi} (r\sin\theta)(-\beta_1 r^2\sin\theta)\beta_2\,r\,d\theta = -\pi r^4\beta_1\beta_2$$

$$c_{34} = \int_0^{2\pi} (-r\cos\theta)(-\beta_1 r^2\sin\theta)\beta_2\,r\,d\theta = 0$$

Step 10:

$$Q_1 = T\beta_2/r$$

$$Q_2 = V_x r\beta_2$$

$$Q_3 = -V_y r\beta_2$$

Step 11:

$$\epsilon_0 = a_{11}(P - Q_1) = [P - (T\beta_2/r)]/2\pi r\beta_1$$

$$K_x = -a_{22}(M_x + Q_2) = (M_x + V_x r\beta_2)/\pi r^3\beta_1$$

$$K_y = -a_{33}(M_y + Q_3) = (M_y - V_y r\beta_2)/\pi r^3\beta_1$$

Step 12:

$$N = \beta_1\left[\frac{P - (T\beta_2/r)}{2\pi r\beta_1} - \left(\frac{M_x + V_x r\beta_2}{\pi r^3\beta_1}\right) r \sin\theta + \left(\frac{M_y - V_y r\beta_2}{\pi r^3\beta_1}\right) r \cos\theta\right] + \beta_2\left[\frac{T}{2\pi r^2} + \frac{V_y \cos\theta + V_x \sin\theta}{\pi r}\right] = \frac{P}{2\pi r} - \frac{M_x \sin\theta}{\pi r^2} + \frac{M_y \cos\theta}{\pi r^2}$$

Step 13:

$$\epsilon = \alpha_1\left(\frac{P}{2\pi r} - \frac{M_x \sin\theta}{\pi r^2} + \frac{M_y \cos\theta}{\pi r^2}\right) + \alpha_2\left(\frac{T}{2\pi r^2} + \frac{V_y \cos\theta}{\pi r} + \frac{V_x \sin\theta}{\pi r}\right)$$

Step 14:

$$\gamma = \alpha_2\left(\frac{P}{2\pi r} - \frac{M_x \sin\theta}{\pi r^2} + \frac{M_y \cos\theta}{\pi r^2}\right) + \alpha_4\left(\frac{T}{2\pi r^2} + \frac{V_y \cos\theta}{\pi r} + \frac{V_x \sin\theta}{\pi r}\right)$$

Step 15:

$$\frac{d\phi}{dz} = \frac{1}{2\pi r^2}\left(\alpha_2 P + \alpha_4 \frac{T}{r}\right)$$

From the above results we note that extension-twist coupling occurs when β_2 or α_2 is not zero, i.e., a P produces some $d\phi/dz$, and a T produces some ϵ_0. Furthermore, the amount of the coupling is consistent with the Maxwell-Betti reciprocal theorem, inasmuch as the $d\phi/dz$ per unit of P is $\alpha_2/2\pi r^2$, while the ϵ_0 per unit of T is also $\alpha_2/2\pi r^2$. Additional coupling is indicated by the presence of V_x in the K_x equation and V_y in the K_y equation; this implies that a horizontal shear produces vertical bending, whereas a vertical shear produces sideways bending. We also note that in this case $d\phi/dz$ does not depend on M_x and M_y; therefore, it is constant over the length L. We note finally that $q(s)$ is independent of the values of β_1, β_2, and β_4, which is consistent with the observations made earlier, following Eqs. (24). The same independence will be seen to hold in the remaining three illustrative applications.

B. Inhomogeneous Circular Tube

We now consider the same cantilever with the same loading but assume that the upper and lower halves of the tube are mirror images of each other about the xz plane. In consequence of this physical symmetry, β_1 and β_4 (also α_1 and α_4) will continue to be constant throughout the tube (independent of both s and z), but β_2 (also α_2) will change sign as we pass from the upper half to the lower half at $s = \pi r$ or $\theta = \pi$. To incorporate this symmetry into the analysis, we shall set $\beta_2 = \beta$ in the upper half of the tube and $\beta_2 = -\beta$ in the lower half. Also, recalling that steps 1 through 8 of the calculation procedure do not involve β_2, we conclude at the outset that all of the findings of the previous example up to and including d_1, d_2, d_3, d_4 will remain valid in the present case. Therefore, the calculation procedure for the present example need only start with step 9. The following results are obtained:

Step 9:

$$c_{11} = \int_0^{\pi r} \beta\, ds + \int_{\pi r}^{2\pi r} (-\beta)\, ds = 0$$

$$c_{21} = \int_0^{\pi} (r \sin\theta)\beta r\, d\theta + \int_{\pi}^{2\pi} (r \sin\theta)(-\beta) r\, d\theta = 4\beta r^2$$

$$c_{31} = \int_0^{\pi} (-r \cos\theta)\beta r\, d\theta + \int_{\pi}^{2\pi} (-r \cos\theta)(-\beta) r\, d\theta = 0$$

$$c_{12} = -\int_0^{\pi} (\beta_1 r\theta)\beta r\, d\theta - \int_{\pi}^{2\pi} (\beta_1 r\theta)(-\beta) r\, d\theta = \pi^2 r^2 \beta_1 \beta$$

$$c_{22} = -\int_0^{\pi} (r \sin\theta)(\beta_1 r\theta)\beta r\, d\theta - \int_{\pi}^{2\pi} (r \sin\theta)(\beta_1 r\theta)(-\beta) r\, d\theta = -4\pi r^3 \beta_1 \beta$$

$$c_{32} = -\int_0^{\pi} (-r \cos\theta)(\beta_1 r\theta)\beta r\, d\theta - \int_{\pi}^{2\pi} (-r \cos\theta)(\beta_1 r\theta)(-\beta) r\, d\theta = -4r^3\beta_1\beta$$

$$c_{13} = \int_0^{\pi} \beta_1 r^2(1 - \cos\theta)\beta r\, d\theta + \int_{\pi}^{2\pi} \beta_1 r^2(1 - \cos\theta)(-\beta) r\, d\theta = 0$$

$$c_{23} = \int_0^{\pi} (r \sin\theta)\beta_1 r^2(1 - \cos\theta)\beta r\, d\theta + \int_{\pi}^{2\pi} (r \sin\theta)\beta_1 r^2(1 - \cos\theta)(-\beta) r\, d\theta = 4r^4\beta_1\beta$$

$$c_{33} = \int_0^{\pi} (-r \cos\theta)\beta_1 r^2(1 - \cos\theta)\beta r\, d\theta + \int_{\pi}^{2\pi} (-r \cos\theta)\beta_1 r^2(1 - \cos\theta)(-\beta) r\, d\theta = 0$$

$$c_{14} = \int_0^{\pi} (-\beta_1 r^2 \sin\theta)\beta r\, d\theta + \int_{\pi}^{2\pi} (-\beta_1 r^2 \sin\theta)(-\beta) r\, d\theta = -4r^3\beta_1\beta$$

$$c_{24} = \int_0^{\pi} (r \sin\theta)(-\beta_1 r^2 \sin\theta)\beta r\, d\theta + \int_{\pi}^{2\pi} (r \sin\theta)(-\beta_1 r^2 \sin\theta)(-\beta) r\, d\theta = 0$$

$$c_{34} = \int_0^{\pi} (-r \cos\theta)(-\beta_1 r^2 \sin\theta)\beta r\, d\theta + \int_{\pi}^{2\pi} (-r \cos\theta)(-\beta_1 r^2 \sin\theta)(-\beta) r\, d\theta = 0$$

Step 10:

$$Q_1 = 4\beta V_x/\pi$$

$$Q_2 = 2\beta T/\pi$$

$$Q_3 = 0$$

Step 11:

$$\epsilon_0 = a_{11}(P - Q_1) = \frac{P - (4\beta V_x/\pi)}{2\pi r\beta_1} = \frac{P}{2\pi r\beta_1} - \frac{2\beta V_x}{\pi^2 r\beta_1}$$

$$K_x = -a_{22}(M_x + Q_2) = \frac{M_x + (2\beta T/\pi)}{\pi r^3\beta_1} = \frac{M_x}{\pi r^3\beta_1} + \frac{2\beta T}{\pi^2 r^3\beta_1}$$

$$K_y = -a_{33}(M_y + Q_3) = (M_y/\pi r^3\beta_1)$$

Step 12:

$$N = \frac{P}{2\pi r} - \frac{2\beta V_x}{\pi^2 r} - \left(\frac{M_x}{\pi r^2} + \frac{2\beta T}{\pi^2 r^2}\right)\sin\theta + \frac{M_y \cos\theta}{\pi r^2}$$
$$\pm \beta\left[\frac{T}{2\pi r^2} + \frac{1}{\pi r}(V_y \cos\theta + V_x \sin\theta)\right]$$

where, in the term $\pm\beta$, the upper sign is to be used for the upper half of the beam $(0<\theta<\pi)$ and the lower sign for the lower half $(\pi<\theta<2\pi)$. [The same convention will apply in all subsequent equations where a double sign ($\pm$ or $\mp$) appears before a term.]

Step 13:

$$\epsilon = \frac{1}{\beta_1}N - \frac{\beta_2}{\beta_1}q$$

$$\epsilon = \frac{1}{\beta_1}\left(\frac{P}{2\pi r} - \frac{2\beta V_x}{\pi^2 r} - \frac{M_x \sin\theta}{\pi r^2} - \frac{2\beta T \sin\theta}{\pi^2 r^2} + \frac{M_y \cos\theta}{\pi r^2}\right.$$
$$\left.\pm \frac{\beta T}{2\pi r^2} \pm \frac{\beta V_y \cos\theta}{\pi r} \pm \frac{\beta V_x \sin\theta}{\pi r}\right)$$
$$\mp \frac{\beta}{\beta_1}\left(\frac{T}{2\pi r^2} + \frac{V_y \cos\theta}{\pi r} + \frac{V_x \sin\theta}{\pi r}\right)$$

$$\epsilon = \frac{1}{\beta_1}\left(\frac{P}{2\pi r} - \frac{M_x \sin\theta}{\pi r^2} + \frac{M_y \cos\theta}{\pi r^2}\right) - \frac{\beta}{\beta_1}\left(\frac{2V_x}{\pi^2 r} + \frac{2T \sin\theta}{\pi^2 r^2}\right)$$

Step 14:

$$\gamma = -\beta_2 \epsilon + \beta_4 q = \mp \beta\epsilon + \beta_4 q$$

$$\gamma = \mp\beta\left[\frac{1}{\beta_1}\left(\frac{P}{2\pi r} - \frac{M_x \sin\theta}{\pi r^2} + \frac{M_y \cos\theta}{\pi r^2}\right)\right.$$
$$\left. - \frac{\beta}{\beta_1}\left(\frac{2V_x}{\pi^2 r} + \frac{2T\sin\theta}{\pi^2 r^2}\right)\right]$$
$$+ \beta_4\left[\frac{T}{2\pi r^2} + \frac{1}{\pi r}(V_y \cos\theta + V_x \sin\theta)\right]$$

Step 15:

$$\frac{d\phi}{dz} = \frac{1}{2\pi r^2}\left[\frac{T}{r}\left(\beta_4 + \frac{8}{\pi^2}\frac{\beta^2}{\beta_1}\right) + \frac{4M_x}{\pi r}\frac{\beta}{\beta_1}\right]$$

In these results, we again see agreement with the Maxwell-Betti reciprocal theorem, in that a T produces no ϵ_0, and a P produces no $d\phi/dz$. This also indicates the absence of extension-torsion coupling. Because of the M_x term in the $d\phi/dz$ equation, the rate of twist is not constant over the length L. Although there is no extension-torsion coupling, there is bending-torsion coupling. This is evident from the equations for K_x and $d\phi/dz$. The presence of T in the former shows that a torque will produce vertical bending, whereas the presence of M_x in the latter shows that vertical bending moments will produce twist. Finally, as is to be expected from the observations following Eqs. (24), $q(s)$ is seen to be independent of β_1, β_2, and β_4.

C. Homogeneous Biconvex Tube

We here consider the cantilever shown in Fig. 6, loaded at its free end with shears V_x and V_y, a torque T, a tension P, and bending moments $M_x(L)$ and $M_y(L)$, producing cross-sectional loads V_x, V_y, T, P, $M_x = M_x(L) + (L-z)V_y$, and $M_y = M_y(L) + (L-z)V_x$. The elastic constants β_1, β_2, β_4 are assumed to be independent of both s and z, as though this tube were a flattened version of the circular tube considered in the section on Homogeneous Circular Tube (Sec. A).

The equation for the top and bottom arcs of the cross section is

$$y = \pm h[1-(x/a)^2] \tag{34}$$

where the upper sign applies to the upper surface and the lower sign to the lower surface. By analytic geometry, the perpendicular distance $p(x_0)$, from the origin 0 to a tangent drawn to the upper or lower surface at $x = x_0$, is found to be

$$p(x_0) = \frac{h[1+(x_0/a)^2]}{\sqrt{1+4(h/a)^2(x_0/a)^2}} \tag{35}$$

However, we shall assume that the thickness ratio h/a is small $(h/a \leq 0.1)$, so that this formula can be simplified to

$$p(x_0) = h[1+(x_0/a)^2] \tag{36}$$

The thinness of the cross section also justifies the following approximations:

$$x(s) = \begin{cases} s-a & \text{for points in the upper surface} \\ 3a-s & \text{for points in the lower surface} \end{cases} \tag{37}$$

$$s(x) = \begin{cases} a+x & \text{for points in the upper surface} \\ 3a-x & \text{for points in the lower surface} \end{cases} \tag{38}$$

$$ds = \begin{cases} +dx & \text{in the upper surface} \\ -dx & \text{in the lower surface} \end{cases} \tag{39}$$

and permits the upper surface to be defined by $0<s<2a$, the lower surface by $2a<s<4a$, and the perimeter to be taken as $4a$.

In the above, we have encountered a quantity y having different formulas for points in the upper surface and points in the lower surface. In what follows, the same situation will arise in connection with other quantities. In order to make clear for which surface a particular symbol is intended, we shall append a superscript t or b (for top and bottom) to the

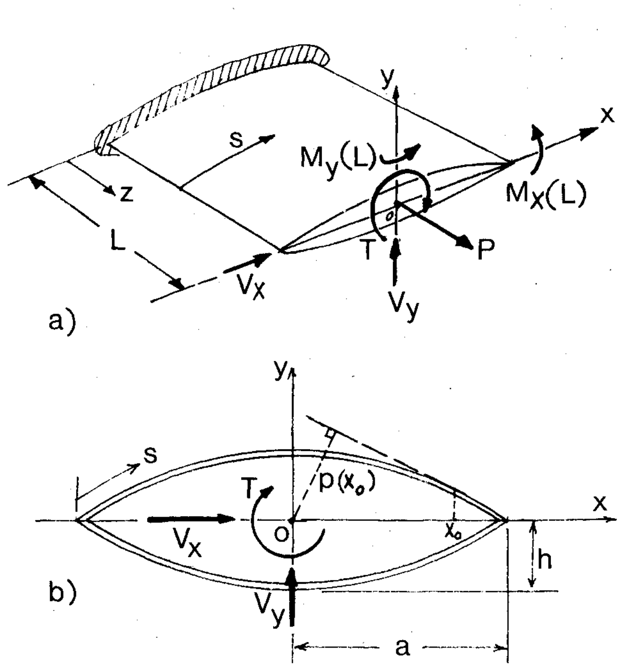

Fig. 6 Biconvex tube: a) loading, b) cross section.

symbol, thereby also making it clear which formula may replace the symbol. By this convention it follows, for example, that $y^t = +h[1-(x/a)^2]$ and $y^b = -h[1-(x/a)^2]$. On the other hand, since the formula given above for $p(x_0)$ applies equally well to both surfaces, superscripts t and b will never be required on the symbol p.

In executing the calculation steps listed at the end of Sec. IV, it often will be necessary to evaluate integrals of the form

$$f(s) = \int_0^s g(s)\, \mathrm{d}s \tag{40}$$

and to have the result as a function of x rather than s. This can be accomplished as follows when the upper limit in Eq. (40) corresponds to a point in the top surface:

$$f(s) = f(s)^t = \int_{-a}^{x} g(x)^t\, \mathrm{d}x = f(x)^t \tag{41}$$

where $g(x)^t$ is understood to mean $g(s)^t$ expressed in terms of x, and $f(x)^t$ also is to be understood as $f(s)^t$ expressed in terms of x. Similarly, when the upper limit in Eq. (40) corresponds to a point in the bottom surface, then

$$f(s) = f(s)^b = \int_{-a}^{a} g(x)^t\, \mathrm{d}x + \int_a^x g(x)^b\,(-\mathrm{d}x) = f(x)^b \tag{42}$$

(Here again, we are sacrificing notational rigor for the sake of simplicity.) A closed line integral of the form

$$f = \oint g(s)\, \mathrm{d}s \tag{43}$$

can be evaluated by replacing the upper limit x in Eq. (42) by $-a$. Thus,

$$f = \int_{-a}^{a} g(x)^t\, \mathrm{d}x + \int_a^{-a} g(x)^b(-\mathrm{d}x)$$
$$= \int_{-a}^{a} [g(x)^t + g(x)^b]\, \mathrm{d}x \tag{44}$$

We will now proceed with the calculation procedure, employing the notation and coordinate transformation already described, making frequent use of Eqs. (42) and (44), and occasionally taking advantage of symmetry to shorten the work. The results are given below. Whenever two formulas are given for a quantity, the upper formula will apply to points in the upper surface and the lower formula to points in the lower surface. For the sake of brevity, the formula for γ, obtained from step 14, is omitted.

Step 1:

$$a_1(s) = \beta_1 s = \begin{cases} \beta_1\,(a+x) \equiv a_1\,(x)^t \\ \beta_1\,(3a-x) \equiv a_1\,(x)^b \end{cases}$$

$$a_2(s) = \beta_1 \int_{-a}^{x} h[1-(x/a)^2]\, \mathrm{d}x$$
$$= \beta_1 h\left(\frac{2a}{3} + x - \frac{x^3}{3a^2}\right) \equiv a_2\,(x)$$

$$a_3(s) = \begin{cases} \beta_1 \int_{-a}^{x} x\, \mathrm{d}x = \frac{1}{2}\beta_1\,(x^2-a^2) \equiv a_3\,(x)^t \\ \beta_1 \int_{-a}^{a} x\, \mathrm{d}x + \beta_1 \int_a^x x(-\mathrm{d}x) \\ = \frac{1}{2}\beta_1\,(a^2-x^2) \equiv a_3\,(x)^b \end{cases}$$

Step 2:

$$b_{11} = \beta_1 \oint \mathrm{d}s = 4a\beta_1$$

$$b_{22} = -4\beta_1 \int_0^a h^2[1-(x/a)^2]^2\, \mathrm{d}x = -\frac{32}{15}\beta_1 h^2 a$$

$$b_{33} = -4\beta_1 \int_0^a x^2\, \mathrm{d}x = -\frac{4}{3}\beta_1 a^3$$

$$b_{ij} = 0 \qquad \text{for } i \neq j$$

Step 3:

$$a_{11} = (4a\beta_1)^{-1}$$

$$a_{22} = \left(-\frac{32}{15}\beta_1 h^2 a\right)^{-1}$$

$$a_{33} = \left(-\frac{4}{3}\beta_1 a^3\right)^{-1}$$

$$a_{ij} = 0 \qquad \text{if } i \neq j$$

Step 4:

$$A = 4\int_0^a y^t\, \mathrm{d}x = 4\int_0^a h[1-(x/a)^2]\, \mathrm{d}x = \frac{8}{3}ha$$

Step 5:

$$a_4 = \int_{-a}^{a} p(x)\,[a_1\,(x)^t + a_1\,(x)^b]\, \mathrm{d}x$$
$$= \int_{-a}^{a} h[1+(x/a)^2](4a\beta_1)\, \mathrm{d}x = \frac{32}{3} ha^2\beta_1$$

$$a_5 = 2\int_{-a}^{a} p(x) a_2\,(x)\, \mathrm{d}x$$
$$= 2\int_{-a}^{a} h[1+(x/a)^2]\beta_1\, h\left(\frac{2a}{3} + x - \frac{x^3}{3a^2}\right) \mathrm{d}x$$
$$= \frac{32}{9} h^2 a^2 \beta_1$$

$$a_6 = \int_{-a}^{a} p(x)\,[a_3\,(x)^t + a_3\,(x)^b]\, \mathrm{d}x = 0$$

Step 6:

$$\epsilon_0' = 0$$

$$K_x' = -\frac{15V_y}{32\beta_1 h^2 a}$$

$$K_y' = -\frac{3V_x}{4\beta_1 a^3}$$

Step 7:

$$q_0 = \frac{3}{16ha}\left(T + \frac{15V_y}{32\beta_1 h^2 a}\,\frac{32h^2a^2\beta_1}{9}\right) = \frac{3T}{16ha} + \frac{5V_y}{16h}$$

$$q(s) = \frac{3T}{16ha} + \frac{5V_y}{16h} - \frac{15V_y}{32\beta_1 h^2 a}\left[\beta_1 h\left(\frac{2a}{3} + x - \frac{x^3}{3a^2}\right)\right]$$
$$- \frac{3V_x}{4\beta_1 a^3}\left[\pm\frac{1}{2}\beta_1\,(x^2-a^2)\right]$$
$$= \frac{3T}{16ha} + \frac{15V_y}{32h}\left(\frac{x^3}{3a^3} - \frac{x}{a}\right) + \frac{3V_x}{8a}\frac{y}{h}$$

Step 8:

$$d_1 = 4a\beta_4$$

$$d_2 = -\beta_4\int_{-a}^{a} [a_1\,(x)^t + a_1\,(x)^b]\,\mathrm{d}x$$

$$= -\beta_4\int_{-a}^{a} 4a\beta_1 \mathrm{d}x = -8a^2\beta_1\beta_4$$

$$d_3 = 2\beta_4\int_{-a}^{a} \beta_1 h\left(\frac{2a}{3} + x - \frac{x^3}{3a^2}\right)\mathrm{d}x = \frac{8}{3}\,a^2h\beta_1\beta_4$$

$$d_4 = \beta_4\int_{-a}^{a} [a_3\,(x)^t + a_3\,(x)^b]\,\mathrm{d}x = 0$$

Step 9:

$$c_{11} = \oint \beta_2\,\mathrm{d}s = 4a\beta_2$$

$$c_{21} = \oint y\beta_2\,\mathrm{d}s = 0$$

$$c_{31} = \oint x\beta_2\,\mathrm{d}s = 0$$

$$c_{12} = -\beta_2\int_{-a}^{a} [a_1\,(x)^t + a_1\,(x)^b]\,\mathrm{d}x$$

$$= -\beta_2\int_{-a}^{a} 4a\beta_1\,\mathrm{d}s = -8a^2\beta_1\beta_2$$

$$c_{22} = -\beta_2\int_{-a}^{a} [y(x)^t\,a_1\,(x)^t + y(x)^b a_1\,(x)^b]\,\mathrm{d}x$$

$$= -\beta_2\int_{-a}^{a}\Big[h\left(1-\frac{x^2}{a^2}\right)(a+x)\beta_1$$

$$-h\left(1-\frac{x^2}{a^2}\right)(3a-x)\,\beta_1\Big]\,\mathrm{d}x = \frac{8}{3}\,a^2h\beta_1\beta_2$$

$$c_{32} = -\beta_2\int_{-a}^{a} x[a_1\,(x)^t + a_1\,(x)^b]\,\mathrm{d}x$$

$$= -\beta_2\int_{-a}^{a} x(4a\beta_1)\,\mathrm{d}x = 0$$

$$c_{13} = 2\beta_2\int_{-a}^{a} a_2\,(x)\,\mathrm{d}x = 2\beta_2\int_{-a}^{a}\beta_1 h\left(\frac{2a}{3} + x - \frac{x^3}{3a^2}\right)\mathrm{d}x$$

$$= \frac{8}{3}\,a^2h\beta_1\beta_2$$

$$c_{23} = \beta_2\int_{-a}^{a} [y(x)^t + y(x)^b]a_2(x)\,\mathrm{d}x = 0$$

$$c_{33} = 2\beta_2\int_{-a}^{a} xa_2\,(x)\,\mathrm{d}x$$

$$= 2\beta_2\int_{-a}^{a} x\beta_1 h\left(\frac{2a}{3} + x - \frac{x^3}{3a^2}\right)\mathrm{d}x$$

$$= \frac{16}{15}\,a^3h\beta_1\beta_2$$

$$c_{14} = \beta_2\int_{-a}^{a} [a_3\,(x)^t + a_3\,(x)^b]\,\mathrm{d}x = 0$$

$$c_{24} = \beta_2\int_{-a}^{a} [y(x)^t a_3\,(x)^t + y(x)^b a_3\,(x)^b]\,\mathrm{d}x$$

$$= 2\beta_2\int_{-a}^{a} h\left(1-\frac{x^2}{a^2}\right)\frac{(x^2-a^2)}{2}\,\beta_1\,\mathrm{d}x = -\frac{16}{15}\,a^3h\beta_1\beta_2$$

$$c_{34} = \beta_2\int_{-a}^{a} x[a_3\,(x)^t + a_3\,(x)^b]\,\mathrm{d}x = 0$$

Step 10:

$$Q_1 = 3T\beta_2/4h$$

$$Q_2 = 4V_x h\beta_2/5$$

$$Q_3 = -V_y\beta_2 a^2/2h$$

Step 11:

$$\epsilon_0 = (4a\beta_1)^{-1}\left(P - \frac{3T\beta_2}{4h}\right) = \frac{P}{4a\beta_1} - \frac{3T}{16ah}\frac{\beta_2}{\beta_1}$$

$$K_x = \left(\frac{32}{15}\,\beta_1 h^2 a\right)^{-1}\left(M_x + \frac{4V_x h\beta_2}{5}\right) = \frac{15M_x}{32ah^2\beta_1} + \frac{3V_x}{8ah}\frac{\beta_2}{\beta_1}$$

$$K_y = \left(\frac{4}{3}\,\beta_1 a^3\right)^{-1}\left(M_y - \frac{V_y a^2\beta_2}{2h}\right) = \frac{3M_y}{4a^3\beta_1} - \frac{3V_y}{8ah}\frac{\beta_2}{\beta_1}$$

Step 12:

$$N = \beta_1\left[\frac{P}{4a\beta_1} - \frac{3T}{16ah}\frac{\beta_2}{\beta_1} - y\left(\frac{15M_x}{32ah^2\beta_1} + \frac{3V_x}{8ah}\frac{\beta_2}{\beta_1}\right)\right.$$

$$\left. - x\left(\frac{3M_y}{4a^3\beta_1} - \frac{3V_y}{8ah}\frac{\beta_2}{\beta_1}\right)\right]$$

$$+ \beta_2\left[\frac{3T}{16ha} + \frac{15V_y}{32h}\left(\frac{x^3}{3a^3} - \frac{x}{a}\right) + \frac{3V_x}{8a}\frac{y}{h}\right]$$

$$= \frac{P}{4a} - \left(\frac{15M_x}{32ah^2}\right)y - \left(\frac{3M_y}{4a^3}\right)x + \beta_2\frac{V_y}{h}\left(\frac{5x^3}{32a^3} - \frac{3x}{32a}\right)$$

Step 13:

$$\epsilon = \epsilon_0 - yK_x - xK_y = \left(\frac{P}{4a\beta_1} - \frac{3T}{16ah}\frac{\beta_2}{\beta_1}\right)$$

$$- y\left(\frac{15M_x}{32ah^2\beta_1} + \frac{3V_x}{8ah}\frac{\beta_2}{\beta_1}\right) - x\left(\frac{3M_y}{4a^3\beta_1} - \frac{3V_y}{8ah}\frac{\beta_2}{\beta_1}\right)$$

Step 15:

$$\frac{\mathrm{d}\phi}{\mathrm{d}z} = \frac{3}{16ha}\left(\alpha_2 P + \frac{3}{4}\,\alpha_4\,\frac{T}{h}\right)$$

We note that the rate of twist $\mathrm{d}\phi/\mathrm{d}z$ is constant. Also, in steps 11 and 15 we again see results that are consistent with the reciprocal theorem, in that the ϵ_0 per unit of T is equal to the $\mathrm{d}\phi/\mathrm{d}z$ per unit of P.

D. Inhomogeneous Biconvex Tube

We consider the same tube as in the preceding section but assume that the upper and lower halves are mirror images of each other, so that β_2 changes sign at $s = 2a$. Therefore, β_2 will be designated as β in the upper half and as $-\beta$ in the lower half. We also recall that steps 1 through 8 of the calculation procedure do not involve β_2; consequently, the results obtained in the previous section by those steps are valid in the present section. Starting with step 9 and again making use of Eq. (44), we obtain the following new results:

Step 9:

$$c_{11} = 0$$

$$c_{21} = \int_{-a}^{a} [y^t\beta + y^b \cdot (-\beta)]\, dx = 2\beta\int_{-a}^{a} y^t\, dx$$

$$= \beta A = \frac{8}{3} ha\beta$$

$$c_{31} = \int_{-a}^{a} [x\beta + x \cdot (-\beta)]\, dx = 0$$

$$c_{12} = -\int_{-a}^{a} [a_1(s)^t\beta + a_1(s)^b \cdot (-\beta)]\, dx = 4a^2\beta_1\beta$$

$$c_{22} = -\int_{-a}^{a} [y^t a_1(s)^t\beta - y^b a_1(s)^b\beta]\, dx = -\frac{16}{3} a^2 h\beta_1\beta$$

$$c_{32} = -\int_{-a}^{a} x[a_1(s)^t\beta - a_1(s)^b\beta]\, dx = -\frac{4}{3} a^3\beta_1\beta$$

$$c_{13} = \int_{-a}^{a} [a_2(s)\beta + a_2(s)(-\beta)]\, dx = 0$$

$$c_{23} = \int_{-a}^{a} [y^t a_2(s)\beta + y^b a_2(s)(-\beta)]\, dx$$

$$= 2\beta\int_{-a}^{a} y^t a_2(s)\, dx = \frac{16}{9} h^2 a^2\beta_1\beta$$

$$c_{33} = \int_{-a}^{a} x a_2(s)[\beta + (-\beta)]\, dx = 0$$

$$c_{14} = \int_{-a}^{a} [a_3(s)^t\beta + a_3(s)^b(-\beta)]\, dx$$

$$= 2\beta\int_{-a}^{a} a_3(s)^t\, dx = -\frac{4}{3} a^3\beta_1\beta$$

$$c_{24} = \int_{-a}^{a} [y^t a_3(s)^t\beta + y^b a_3(s)^b(-\beta)]\, dx = 0$$

$$c_{34} = \int_{-a}^{a} x[a_3(s)^t\beta + a_3(s)^b(-\beta)]\, dx$$

$$= 2\beta\int_{-a}^{a} x a_3(s)^t\, dx = 0$$

Step 10:

$$Q_1 = \beta V_x$$

$$Q_2 = \beta T/2$$

$$Q_3 = 0$$

Step 11:

$$\epsilon_0 = \frac{1}{4a\beta_1}(P - \beta V_x)$$

$$K_x = \frac{15}{32\beta_1 h^2 a}\left(M_x + \frac{\beta T}{2}\right)$$

$$K_y = \frac{3M_y}{4\beta_1 a^3}$$

Step 12:

$$N = \frac{1}{4a}(P - \beta V_x) - \frac{15y}{32h^2 a}\left(M_x + \frac{\beta T}{2}\right) - \frac{3xM_y}{4a^3}$$
$$\pm \beta\left[\frac{3T}{16ha} + \frac{15V_y}{32h}\left(\frac{x^3}{3a^3} - \frac{x}{a}\right) + \frac{3V_x}{8a}\frac{y}{h}\right]$$

(Here and below, where $\pm$ appears, the upper sign applies to the upper surface and the lower sign to the lower surface.)

Step 13:

$$\epsilon = \frac{1}{4a\beta_1}(P - \beta V_x) - \frac{15y}{32\beta_1 h^2 a}\left(M_x + \frac{\beta T}{2}\right) - \frac{3xM_y}{4\beta_1 a^3}$$

Step 14:

$$\gamma = \pm\frac{\beta}{\beta_1}\left[\frac{15y}{32h^2 a}\left(M_x + \frac{\beta T}{2}\right) + \frac{3xM_y}{4a^3} - \frac{1}{4a}(P - \beta V_x)\right]$$
$$+ \beta_4\left[\frac{3T}{16ha} + \frac{15V_y}{32h}\left(\frac{x^3}{3a^3} - \frac{x}{a}\right) + \frac{3V_x}{8a}\frac{y}{h}\right]$$

Step 15:

$$\frac{d\phi}{dz} = \frac{3}{16ha}\left[\frac{5\beta}{4\beta_1 h}\left(M_x + \frac{\beta T}{2}\right) + \frac{3}{4}\frac{\beta_4 T}{h}\right]$$

We note in this case, because of the presence of M_x in the $d\phi/dz$ equation, the rate of twist will vary with z if there is any vertical shear V_y in the loading.

VI. Concluding Remarks

A theory has been presented for computing the shear flows, cross-sectional normal stresses, and rate of twist in linearly elastic single-cell thin-walled beams with anisotropic walls, The assumed anisotropy is such as to allow coupling between cross-sectional shear flow and longitudinal strain and, reciprocally, between shear strain and longitudinal stress. The beam is assumed to be of uniform cross section, with wall elastic constants that are constant longitudinally but may vary around the periphery of the cross section. The loads and reactions are assumed to be applied to the end cross sections only but otherwise are quite general. They may include longitudinal thrust, shears, a twisting moment, and bending moments. If, in an actual case, the beam is nonuniform and carries distributed loads, but can be approximated as a succession of uniform segments loaded only at their junctions, then the present theory will apply to the individual segments.

Essential features of the present theory are the assumptions that 1) the shape of the cross section is preserved, and 2) the longitudinal strains vary linearly over the cross section. Thus, it is a generalization of the familiar classical theory of isotropic (or specially orthotropic) thin-walled beams. It is shown that anisotropy of the walls can, as expected, lead to coupling phenomena not present in the classical theory, e.g., twist due to bending moments, bending due to torque, twist due to tension, and extension due to torque, thus confirming the well-known possibility of tailoring the elastic behavior of thin-walled beams through the use of laminated composites. It is also shown that anisotropy of the walls can lead to a nonconstant rate of twist even if the cross-sectional torque and shears are constant along the length. Another interesting outcome of the present theory is that the shear flows are independent of the coupling elastic constant (α_2 or β_2). (This and other interesting predictions of the present theory regarding the nature of the shear flows are confirmed to a high degree of accuracy in Refs. 7 and 8.)

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